

A discontinuous Galerkin method for time fractional diffusion equations with variable coefficients

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November 3, 2015

Abstract We propose a piecewise-linear, time-stepping discontinuous Galerkin method to solve numerically a time fractional diffusion equation involving Caputo derivative of order $\mu \in (0, 1)$ with variable coefficients. For the spatial discretization, we apply the standard piecewise linear continuous Galerkin method. Well-posedness of the fully discrete scheme and error analysis will be shown. For a time interval $(0, T)$ and a spatial domain Ω , our analysis suggest that the error in $L^2((0, T), L^2(\Omega))$ -norm is of order $O(k^{2-\frac{\mu}{2}} + h^2)$ (that is, short by order $\frac{\mu}{2}$ from being optimal in time) where k denotes the maximum time step, and h is the maximum diameter of the elements of the (quasi-uniform) spatial mesh. However, our numerical experiments indicate optimal $O(k^2 + h^2)$ error bound in the stronger $L^\infty((0, T), L^2(\Omega))$ -norm. Variable time steps are used to compensate the singularity of the continuous solution near $t = 0$.

Keywords Fractional diffusion, variable coefficients, discontinuous Galerkin method, convergence analysis

1 Introduction

In this paper, we investigate a numerical solution that allows a time discontinuity for solving time fractional diffusion equations with variable diffusivity. Let Ω be a bounded convex polygonal domain in \mathbb{R}^d ($d = 1, 2, 3$), with a boundary $\partial\Omega$, and

The support of the Science Technology Unit at KFUPM through King Abdulaziz City for Science and Technology (KACST) under National Science, Technology and Innovation Plan (NSTIP) project No. 13-MAT1847-04 is gratefully acknowledged.

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$T > 0$ be a fixed time. Then the fractional model problem is given by:

$$\begin{aligned} {}^c D^\mu u(x, t) - \nabla \cdot (\mathcal{A}(x, t) \nabla u(x, t)) &= f(x, t) & \text{on } \Omega \times (0, T], \\ u(x, 0) &= u_0(x) & \text{on } \Omega, \\ u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T], \end{aligned} \quad (1)$$

where we assume that $\mathcal{A} \in \mathcal{C}^1([0, T], L^\infty(\Omega))$ and satisfies

$$0 < a_{\min} < \mathcal{A}(x, t) < a_{\max} < \infty \quad \text{on } \overline{\Omega} \times [0, T]. \quad (2)$$

Here, ${}^c D^\mu$ is the Caputo's fractional derivative defined by

$${}^c D^\mu v(t) = I^{1-\mu} v'(t) := \int_0^t \omega_{1-\mu}(t-s) v'(s) ds \quad \text{with} \quad \omega_{1-\mu}(t) := \frac{t^{-\mu}}{\Gamma(1-\mu)},$$

where throughout the paper, $0 < \mu < 1$. Noting that, $I^{1-\mu}$ is the Riemann Liouville fractional integral operator, and v' denotes the time partial derivative of v .

Over the past few decades, researchers have observed numerous biological, physical and financial systems in which some key underlying random motion conform to a model where the diffusion is anomalously slow (subdiffusion) and not to the classical model of diffusion. For instance, the fractional diffusion model problem (1) is known to capture well the dynamics of subdiffusion processes, in which the mean square variance grows at a rate slower than that in a Gaussian process, see [31]. Fractional diffusion has been successfully used to describe diffusion in media with fractal geometry [30], highly heterogeneous aquifer [1], and underground environmental problem [12]. Two distinct approaches can be used for modelling fractional sub-diffusion. One based on fractional Brownian motion and Langevin equations [17, 34], this leads to a diffusion equation with a varying diffusion coefficient exhibiting a fractional power law scaling in time [34]. The other based on continuous time random walks and master equations with power law waiting time densities which leads to a diffusion equation with fractional order temporal derivatives operating on the spatial Laplacian [29].

The innovation of this paper is to investigate the piecewise linear time-stepping discontinuous Galerkin (DG) method combined with the standard finite elements (FEs) in space for solving numerically time fractional models with variable diffusion coefficients of the form (1). Since their inception in the early 1970s, DG methods have found numerous applications [5], including for the time discretization of fractional diffusion and fractional wave equations, [22, 23, 24]. Their advantages include excellent stability properties and suitability for adaptive refinement based on a posteriori error estimates [8] to handle problems with low regularity. The present work is motivated by an earlier paper [23]. There in, the first author and McLean considered a piecewise-linear DG method for a fractional diffusion problem with a constant diffusivity:

$$u'(x, t) - {}^R D^{1-\mu} \Delta u(x, t) = f(x, t) \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (3)$$

where ${}^R D^{1-\mu} u := \frac{\partial}{\partial t} (I^\mu u)$ (Riemann–Liouville fractional derivative). Recently, high order hp -DG methods with exponential rates of convergence for fractional diffusion (3) and also for fractional wave equations were studied in [25, 28]. Noting

that, when \mathcal{A} is constant and $f \equiv 0$ in (1), one may look at (3) as an alternative representation of (1).

Numerical solutions for model problems of the form (1) with constant diffusion parameter \mathcal{A} have attracted considerable interest in recent years. The case of variable coefficients is indeed very interesting and also practically important. However, due to the additional difficulty in this case, there are only few papers in the existing literature. With $\Omega = (0, L)$, Alikhanov [2] constructed a new difference analog of the time fractional Caputo derivative with the order of approximation $O(k^{3-\mu})$. Difference schemes of order $O(h^q + k^2)$ (with $q \in \{2, 4\}$) were proposed and analyzed assuming that u is sufficiently regular, where k is the temporal grid size and h is the spatial grid size. For a time independent diffusivity, Zhao and Xu [39] proposed a compact difference scheme for (1). Stability and convergence properties of the scheme were proved. For time fractional convection-diffusion problems, Cui [7] studied a compact exponential scheme. The stability and the convergence analysis were showed assuming that the coefficients of the model problem are constants. For time independent coefficients, Saadatmandi et al. [33] investigated the Sinc-Legendre collocation method.

For *one-dimensional* spatial domains and constant diffusion parameter \mathcal{A} , Murio [21] and Zhang et al. [36] studied two classes of finite difference (FD) methods. Stability properties were provided. Another FD scheme in time (with $L1$ approximation for the Caputo fractional derivative) combined with the spatial fourth order compact difference approach was studied by Ren et al. [32]. Convergence rates of order $O(k^{1+\mu} + h^4)$ were proved. Murillo and Yuste [20] presented an implicit FD method over non-uniform time steps. An adaptive procedure was described to choose the size of the time meshes. Lin and Xu [16] combined a FD scheme in time and a spectral method in space. Accuracy of order $O(k^{1+\mu} + r^{-m})$ was proved, where r is the spatial polynomial degree, and m is related to the regularity of the exact solution u . Later, Li and Xu [15] developed and analyzed a time-space spectral method. Zhao and Sun [38] combined an order reduction approach and $L1$ discretization of the fractional derivative. A box-type scheme was constructed and a convergence rate of order $O(k^{1+\mu} + h^2)$ had been proved. Finite central differences in time combined with the FE method in space was studied by Li and Xu [14]. For a smooth u , a convergence rate of order $O(k^2 + h^{\ell+1})$ was achieved where ℓ is the degree of the FE solutions in space. Recently, a similar convergence rate was shown by Zeng et al. [40] where the fractional linear multistep method was used for the time discretization. For a high-order local DG method for space discretization, we refer to the work by Xu and Zheng [35].

For *two- or three-dimensional* spatial domains with $\mathcal{A} = 1$ in (1), Brunner et al. [3] used an algorithm that couples an adaptive time stepping and adaptive spatial basis selection approach for the numerical solution of (1). A semi-discrete piecewise linear Galerkin FE and lumped mass Galerkin methods were studied by Jin et al. [13]. An optimal error with respect to the regularity error estimates was established for $f \equiv 0$ and non-smooth initial data u_0 . Cui [6] studied the convergence analysis of compact alternating direction implicit (ADI) schemes for sufficiently smooth solutions of (1). For *three-dimensional* spatial domains, a fractional ADI scheme was proposed and analyzed by Chen et al. [4]. Mustapha et al. [26] proposed *low-high* order time stepping discontinuous Petrov-Galerkin methods combined with FEs in space. Using variable time meshes, $O(k^{m+(1-\mu)/2} + h^{r+1})$ convergence rates were shown, where m and r are the degrees of approximate solutions in the time

and spatial variables, respectively. Optimal convergence rates in both variables were demonstrated numerically. In [27], a hybridizable DG method in space was extensively studied by Mustapha et al..

The outline of the paper is as follows. Section 2 introduces a fully discrete DG FE scheme. In Section 3, we prove the stability of the discrete solution and provide a remark about the existence and uniqueness of the numerical solution. Section 4 is devoted to introduce time and space projection operators that will be used later to show the convergence of the numerical scheme. The error analysis is given in Section 5. Using suitable refined time-steps (towards $t = 0$) and quasi-uniform spatial meshes, in the $L^2((0, T), L^2(\Omega))$ -norm, convergence of order $O(k^{2-\frac{\mu}{2}} + h^2)$ is achieved. Section 6 is dedicated to present a sample of numerical test which illustrate that our error bounds are pessimistic. For a strongly graded time mesh, in the stronger $L^\infty((0, T), L^2(\Omega))$ -norm, we observe optimal convergence rates, that is, error of order $O(k^2 + h^2)$.

2 The numerical method

To describe our fully discrete DG FE method, we introduce a time partition of the interval $[0, T]$ given by the points: $0 = t_0 < t_1 < \dots < t_N = T$. We set $I_n = (t_{n-1}, t_n)$ and $k_n = t_n - t_{n-1}$ for $1 \leq n \leq N$ with $k := \max_{1 \leq n \leq N} k_n$. Let $S_h \subseteq H_0^1(\Omega)$ denotes the space of continuous, piecewise polynomials of total degree ≤ 1 with respect to a quasi-uniform partition of Ω into conforming triangular finite elements, with maximum diameter h . Next, we introduce our time-space finite dimensional DG FE space:

$$\mathcal{W} = \{w \in L^2((0, T), S_h) : w|_{I_n} \in \mathcal{P}_1(S_h) \text{ for } 1 \leq n \leq N\}$$

where $\mathcal{P}_1(S_h)$ denotes the space of linear polynomials in the time variable t , with coefficients in S_h . We denote the left-hand limit, right-hand limit and jump at t_n by

$$w^n := w(t_n) = w(t_n^-), \quad w_+^n := w(t_n^+), \quad [w]^n := w_+^n - w^n,$$

respectively. The weak form of the fractional diffusion equation in (1) is

$$\int_{I_n} [\langle {}^c D^\mu u, v \rangle + a(t, u, v)] dt = \int_{I_n} \langle f, v \rangle dt, \quad \forall v \in L^2(I_n, H^1(\Omega)). \quad (4)$$

Throughout the paper, $\langle \cdot, \cdot \rangle$ denotes the L^2 -inner product and $\|\cdot\|$ is the associated norm, and $\|\cdot\|_m$ ($m \geq 1$) denotes the norm on the Sobolev space $H^m(\Omega)$. We use $\|\cdot\|_{L^q(Y)}$ ($q \geq 1$) to denote the norm on $L^q((0, T), Y(\Omega))$ for any Sobolev space $Y(\Omega)$.

For each fixed $t \in (0, T]$, $a(t, \cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ is the bilinear form

$$a(t, v, w) = \langle \mathcal{A}(\cdot, t) \nabla v, \nabla w \rangle = \int_{\Omega} \mathcal{A}(x, t) \nabla v(x) \cdot \nabla w(x) dx$$

associated with the operator $\nabla \cdot (\mathcal{A}(\cdot, t) \nabla \cdot)$ which is symmetric and positive definite (by (2)), that is, there exist positive constants c_0 and c_1 such that

$$c_0 \|v(t)\|_1^2 \leq |v(t)|_1^2 := a(t, v, v) \leq c_1 \|v(t)\|_1^2 \quad \forall v(t) \in H_0^1(\Omega). \quad (5)$$

The DG FE approximation $U \in \mathcal{W}$ is defined as follows: Given $U(t)$ for $0 \leq t \leq t_{n-1}$, the solution $U \in \mathcal{P}_1(S_h)$ on I_n is determined by requesting that for $1 \leq n \leq N$,

$$\int_{I_n} [\langle {}^c D_{dg}^\mu U + \sum_{j=0}^{n-1} \omega_{1-\mu}(t-t_j) [U]^j, X \rangle + a(t, U, X)] dt = \int_{I_n} \langle f, X \rangle dt, \quad \forall X \in \mathcal{P}_1(S_h),$$

with $U_+^0 = U^0 \in S_h$ is a suitable approximation of the initial data u_0 , where

$${}^c D_{dg}^\mu U(t) := \sum_{j=1}^n \int_{t_{j-1}}^{\min\{t_j, t\}} \omega_{1-\mu}(t-s) U'(s) ds \quad \text{for } t \in I_n.$$

Since

$$\begin{aligned} {}^R D^\mu U(t) &:= \frac{\partial}{\partial t} \int_0^t \omega_{1-\mu}(t-s) U(s) ds \\ &= {}^c D_{dg}^\mu U(t) + \omega_{1-\mu}(t) U^0 + \sum_{j=1}^{n-1} \omega_{1-\mu}(t-t_j) [U]^j \quad \text{for } t \in I_n, \end{aligned} \quad (6)$$

our scheme can be rewritten in a compact form as follows: for $1 \leq n \leq N$,

$$\int_{I_n} [\langle {}^R D^\mu U, X \rangle + a(t, U, X)] dt = \int_{I_n} \langle f + \omega_{1-\mu}(t) U^0, X \rangle dt \quad \forall X \in \mathcal{P}_1(S_h). \quad (7)$$

Noting that, since the DG FE scheme (7) amounts to a square linear system, the existence of the numerical solution U follows from its uniqueness. The uniqueness follows immediately from the above stability property in Theorem 1.

3 Stability of the numerical solution

To show the stability of the DG FE scheme (7), we claim first the identity: $v(t) = I^\mu({}^R D^\mu v)(t)$ for any $v \in \mathcal{W}$. If v is an absolutely continuous function in the time variable, this identity follows by applying the fractional integral operator I^μ to both sides of the equality ${}^R D^\mu v(t) = {}^c D^\mu v(t) + \omega_{1-\mu}(t)v(0)$ and then changing the order of integrals and using the identity: $\int_s^t \omega_{1-\mu}(t-s) \omega_\mu(\tilde{t}-t) dt = 1$.

Lemma 1 *If $v \in \mathcal{W}$, then*

$$v(t) = I^\mu({}^R D^\mu v)(t) \quad \text{for } t \in I_n \quad \text{with } 1 \leq n \leq N.$$

Proof Since v has possible discontinuities at the time nodes t_0, t_1, \dots, t_{j-1} , from (6),

$${}^R D^\mu v(s) = \omega_{1-\mu}(s) v_+^0 + \sum_{i=1}^{j-1} \omega_{1-\mu}(s-t_i) [v]^i + {}^c D_{dg}^\mu v(s) \quad \text{for } s \in I_j. \quad (8)$$

Applying the operator I^μ to both sides and using $I^\mu({}^c D^\mu v)(t) = \int_0^t v'(s) ds$, we observe

$$\begin{aligned} I^\mu({}^R D^\mu v)(t) &= v_+^0 + \sum_{j=2}^{n-1} \int_{I_j} \omega_\mu(t-s) \sum_{i=1}^{j-1} \omega_{1-\mu}(s-t_i) [v]^i ds \\ &\quad + \int_{t_{n-1}}^t \omega_\mu(t-s) \sum_{i=1}^{n-1} \omega_{1-\mu}(s-t_i) [v]^i ds + \int_0^t v'(s) ds \quad \text{for } t \in I_n. \end{aligned}$$

Now, changing the order of summations and rearranging the terms yield

$$\begin{aligned} I^\mu({}^R D^\mu v)(t) &= v_+^0 + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \int_{I_j} \omega_\mu(t-s) \omega_{1-\mu}(s-t_i) [v]^i ds \\ &\quad + \sum_{i=1}^{n-1} \int_{t_{n-1}}^t \omega_\mu(t-s) \omega_{1-\mu}(s-t_i) [v]^i ds + \sum_{j=1}^n \int_{t_{j-1}}^{\min\{t, t_j\}} v'(s) ds \\ &= v_+^0 + \sum_{i=1}^{n-2} \int_{t_i}^t \omega_\mu(t-s) \omega_{1-\mu}(s-t_i) [v]^i ds \\ &\quad + \int_{t_{n-1}}^t \omega_\mu(t-s) \omega_{1-\mu}(s-t_{n-1}) [v]^{n-1} ds + \sum_{j=1}^n \int_{t_{j-1}}^{\min\{t, t_j\}} v'(s) ds. \end{aligned}$$

Integrating and simplifying, then we have

$$I^\mu({}^R D^\mu v)(t) = v_+^0 + \sum_{i=1}^{n-1} [v]^i + \sum_{j=1}^{n-1} (v^j - v_+^{j-1}) + v(t) - v_+^{n-1} = v(t) \quad \text{for } t \in I_n. \quad \square$$

In the next lemma we state some important properties of the Riemman Liouville fractional operators. These properties will be used to show the stability of the numerical scheme, as well as, in our error analysis in the forthcoming section.

For $\ell \in \{0, 1\}$, we let $\mathcal{C}^\ell(J_n, L^2(\Omega))$ ($J_n := \cup_{j=1}^n I_j$) denote the space of functions $v : J_n \rightarrow L^2(\Omega)$ such that the restriction $v|_{I_j}$ extends to an ℓ -times continuously differentiable function on the closed interval \bar{I}_j for $1 \leq j \leq n$. For later use, we let

$$\|v\|_{I_j} := \sup_{t \in I_j} \|v(t)\| \quad \text{and} \quad \|v\|_{J_n} := \max_{j=1}^n \|v\|_{I_j},$$

where we drop n when $J_n = J_N$.

Lemma 2 For $1 \leq n \leq N$ and for $0 < \alpha < 1$, we have

(i) The operator ${}^R D^\alpha$ satisfies: for $v \in \mathcal{C}^1(J_n, L^2(\Omega))$,

$$\int_0^{t_n} \langle {}^R D^\alpha v, v \rangle dt \geq \frac{(\pi\alpha)^\alpha}{(1+\alpha)^{1+\alpha}} \cos\left(\frac{\alpha\pi}{2}\right) t_n^{-\alpha} \int_0^{t_n} \|v(t)\|^2 dt.$$

(ii) The integral operator I^α satisfies: for $v, w \in \mathcal{C}^0(J_n, L^2(\Omega))$

$$\left| \int_0^{t_n} \langle I^\alpha v, w \rangle dt \right|^2 \leq \sec^2\left(\frac{\alpha\pi}{2}\right) \int_0^{t_n} \langle I^\alpha v, v \rangle dt \int_0^{t_n} \langle I^\alpha w, w \rangle dt.$$

Proof The property (i) was proven in [18, Theorem A.1] by using the Laplace transform and Plancherel Theorem. For the proof of the property (ii), see [28, Lemma 3.1]. \square

The next theorem shows the stability of the DG FE scheme.

Theorem 1 *Assume that $U^0 \in L^2(\Omega)$ and $f \in L^2((0, T), L^2(\Omega))$. Then,*

$$\int_0^T \|U\|_1^2 dt \leq CT^{1-\mu} \|U^0\|^2 + C \int_0^T \|f\|^2 dt.$$

Proof Choosing $X = U$ in the DG FE scheme (7), and then summing over n , we obtain

$$\int_0^T [\langle {}^R D^\mu U, U \rangle + a(t, U, U)] dt = \int_0^T \langle f + \omega_{1-\mu}(t) U^0, U \rangle dt.$$

Since $a(\cdot, U, U) \geq c_0 \|U\|_1^2$ by (5) and $\langle f, U \rangle \leq \frac{1}{2c_0} \|f\|^2 + \frac{c_0}{2} \|U\|^2$, we have

$$\int_0^T [\langle {}^R D^\mu U, U \rangle + \frac{c_0}{2} \|U\|_1^2] dt \leq \int_0^T \left(\langle \omega_{1-\mu}(t) U^0, U \rangle + \frac{1}{2c_0} \|f\|^2 \right) dt.$$

Using the identity $U(t) = I^\mu ({}^R D^\mu U)(t)$ from Lemma 1, Lemma 2 (ii), the inequality $ab \leq \frac{a^2}{4} + b^2$, and the identity $I^\mu \omega_{1-\mu}(t) = 1$, yield

$$\begin{aligned} \int_0^T \langle \omega_{1-\mu}(t) U^0, U \rangle dt &= \int_0^T \langle \omega_{1-\mu}(t) U^0, I^\mu ({}^R D^\mu U) \rangle dt \\ &\leq \frac{1}{4} \int_0^T \langle {}^R D^\mu U, U \rangle dt + \sec^2(\mu\pi/2) \int_0^T \omega_{1-\mu}(t) (I^\mu \omega_{1-\mu})(t) dt \|U^0\|^2 \\ &\leq \frac{1}{4} \int_0^T \langle {}^R D^\mu U, U \rangle dt + CT^{1-\mu} \|U^0\|^2. \end{aligned} \quad (9)$$

To complete the proof, we combine the above two equations and use the positivity property of the operator ${}^R D^\mu$ given by Lemma 2 (i). \square

4 Projections and errors

In this section, we introduce time and space projections, and then derive some bounds and errors properties that will be used later in our convergence analysis.

4.1 Projection in space

For each $t \in [0, T]$, the elliptic projection operator $R_h : H_0^1(\Omega) \rightarrow S_h$ is defined by

$$a(t, R_h v - v, \chi) = 0 \quad \forall \chi \in S_h. \quad (10)$$

By the assumption $\mathcal{A} \in \mathcal{C}^1([0, T], L^\infty(\Omega))$, for each $t \in (0, T)$, the projection error $\xi := R_h u - u$ has the well-known approximation property:

$$\|\xi(t)\| + h \|\nabla \xi(t)\| \leq C h^2 \|u(t)\|_2 \quad \text{for } u(t) \in H^2(\Omega) \cap H_0^1(\Omega). \quad (11)$$

Moreover, we have

$$\|\xi'(t)\| \leq C h^2 (\|u(t)\|_2 + \|u'(t)\|_2) \quad \text{for } u(t), u'(t) \in H^2(\Omega) \cap H_0^1(\Omega). \quad (12)$$

To show (12), we decompose ξ' as: $\xi' = g_h + (R_h u' - u')$ where for each $t \in (0, T)$, $g_h(t) = (R_h u)'(t) - R_h u'(t) \in S_h$. Since $\|(R_h u' - u')(t)\| \leq C h^2 \|u'(t)\|_2$ by the approximation property (11) applied to u' , it remains to derive a similar bound for g_h . To do so, we use the Nitsches trick: for each $t \in (0, T)$, there is $\phi \in H^2(\Omega)$ such that

$$-\nabla \cdot (\mathcal{A}(t) \nabla \phi) = g_h(t) \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega$$

with $\|\phi\|_2 \leq C \|\nabla \cdot (\mathcal{A}(t) \nabla \phi)\|$ which holds for convex polyhedral domains. Taking the inner product with $g_h(t) \in S_h$, and then using the orthogonality property of R_h ,

$$\|g_h(t)\|^2 = a(t, \phi, g_h(t)) = a(t, R_h \phi, g_h(t)) = a(t, g_h(t), R_h \phi).$$

But, by the definition of R_h , for each $t \in (0, T)$,

$$\begin{aligned} a(t, g_h(t), \chi) &= a(t, (R_h u)'(t), \chi) - a(t, u'(t), \chi) \\ &= \frac{d}{dt} a(t, \xi(t), \chi) - \langle \mathcal{A}'(t) \nabla \xi(t), \nabla \chi \rangle \\ &= -\langle \mathcal{A}'(t) \nabla \xi(t), \nabla \chi \rangle \quad \forall \chi \in S_h. \end{aligned} \quad (13)$$

Therefore,

$$\|g_h(t)\|^2 = -\langle \mathcal{A}'(t) \nabla \xi(t), \nabla R_h \phi \rangle = \langle \mathcal{A}'(t) \nabla \xi(t), \nabla (\phi - R_h \phi) \rangle + \langle \xi(t), \nabla \cdot (\mathcal{A}'(t) \nabla \phi) \rangle.$$

Finally, using the Cauchy-Schwarz inequality, the approximation property in (11), and the inequality $\|\phi\|_2 \leq C \|\nabla \cdot (\mathcal{A}(t) \nabla \phi)\|$, we observe

$$\|g_h(t)\|^2 \leq C \|\nabla(\phi - R_h \phi)\| \|\nabla \xi(t)\| + C \|\xi(t)\| \|\phi\|_2 \leq C h^2 \|g_h(t)\| \|u(t)\|_2.$$

The proof of (12) is completed now.

4.2 Projection in time

The local L^2 -projection operator $\Pi_k : L^2(I_n, L^2(\Omega)) \rightarrow \mathcal{C}(I_n, \mathcal{P}_1(L^2(\Omega)))$ defined by:

$$\int_{I_n} \langle \Pi_k v - v, w \rangle dt = 0 \quad \forall w \in \mathcal{P}_1(L^2(\Omega)) \quad \text{for } 1 \leq n \leq N,$$

where $\mathcal{P}_1(L^2(\Omega))$ is the space of linear polynomials in the time variable t , with coefficients in $L^2(\Omega)$. Explicitly,

$$\Pi_k v(t) = \frac{12}{k_n^3} (t - t_{n-\frac{1}{2}}) \int_{I_n} (s - t_{n-\frac{1}{2}}) v(s) ds + \frac{1}{k_n} \int_{I_n} v(s) ds \quad \text{for } t \in I_n,$$

where $t_{n-\frac{1}{2}} := (t_{n-1} + t_n)/2$. Hence, for $v' \in L^1(I_n, L^2(\Omega))$,

$$(\Pi_k v)'(t) = \frac{12}{k_n^3} \int_{I_n} (s - t_{n-\frac{1}{2}}) v(s) ds = \frac{6}{k_n^3} \int_{I_n} (t_n - t)(s - t_{n-1}) v'(s) ds.$$

Thus, for $1 \leq n \leq N$, we have

$$\| \Pi_k v(t) \| \leq \frac{4}{k_n} \int_{I_n} \| v(s) \| ds \quad \text{and} \quad \| (\Pi_k v)'(t) \| \leq \frac{3}{2k_n} \int_{I_n} \| v'(s) \| ds. \quad (14)$$

Setting $\eta_v = \Pi_k v - v$, we have the well-known projection error bound: for $t \in I_n$,

$$\| \eta_v(t) \| + k_n \| \eta_v'(t) \| \leq C k_n^{\ell-1} \int_{I_n} \left\| \frac{\partial^\ell v}{\partial t^\ell}(s) \right\| ds \quad \text{for } \ell = 1, 2. \quad (15)$$

Next, we show an error bound property of Π_k that involves the operator ${}^R D^\mu$.

Lemma 3 *Let $\frac{\partial^\ell v}{\partial t^\ell} \in L^1((0, t_n), L^2(\Omega))$ for $\ell \in \{1, 2\}$. We have*

$$\int_{I_n} \langle {}^R D^\mu \eta_v, \eta_v \rangle dt \leq C k_n^{1-\mu} \max_{j=1}^n k_j^{2\ell-2} \left(\int_{I_j} \left\| \frac{\partial^\ell v}{\partial t^\ell} \right\| dt \right)^2 \quad \text{for } 1 \leq n \leq N.$$

Proof We integrate by parts and notice that

$$\begin{aligned} \int_{I_n} \langle {}^R D^\mu \eta_v, \eta_v \rangle dt &= \langle I^{1-\mu} \eta_v(t), \eta_v(t) \rangle \Big|_{t_{n-1}^+}^{t_n^-} - \int_{I_n} \langle I^{1-\mu} \eta_v, \eta_v' \rangle dt \\ &= \langle \mathcal{I}^n(t_n), \eta_v(t_n) \rangle - \int_{I_n} \langle \mathcal{I}^n(t), \eta_v'(t) \rangle dt, \end{aligned} \quad (16)$$

where for $t \in I_n$,

$$\begin{aligned} \mathcal{I}^n(t) &:= I^{1-\mu} \eta_v(t) - I^{1-\mu} \eta_v(t_{n-1}) \\ &= \int_0^{t_{n-1}} [\omega_{1-\mu}(t-s) - \omega_{1-\mu}(t_{n-1}-s)] \eta_v(s) ds + \int_{t_{n-1}}^t \omega_{1-\mu}(t-s) \eta_v(s) ds. \end{aligned}$$

Simplifying then integrating, we observe

$$\begin{aligned} \| \mathcal{I}^n(t) \| &\leq \left(\int_0^{t_{n-1}} [\omega_{1-\mu}(t_{n-1}-s) - \omega_{1-\mu}(t-s)] ds + \int_{t_{n-1}}^t \omega_{1-\mu}(t-s) ds \right) \| \eta_v \|_{J_n} \\ &\leq 2 \omega_{2-\mu}(k_n) \| \eta_v \|_{J_n} \quad \text{for } t \in I_n. \end{aligned}$$

Therefore, an application of the Cauchy-Schwarz inequality gives

$$\int_{I_n} | \langle {}^R D^\mu \eta_v, \eta_v \rangle | dt \leq 2 \omega_{2-\mu}(k_n) \| \eta_v \|_{J_n} \left(\| \eta_v(t_n) \| + \int_{I_n} \| \eta_v' \| dt \right),$$

and hence, using the error projection in (15), we obtain the desired bound. \square

Since ${}^R D^\mu v(t) = \omega_{1-\mu}(t)v(0) + I^{1-\mu}v'(t)$ and since $\| \eta_v \|_{I_n} \leq 5 \| v \|_{I_n}$ by the triangle inequality and the first inequality in (14), we have

$$\int_{I_n} | \langle {}^R D^\mu v, \eta_v \rangle | dt \leq 5 \| v \|_{I_n} \int_{I_n} (\omega_{1-\mu}(t) \| v(0) \| + \| I^{1-\mu} v' \|) dt.$$

Summing over n gives

$$\int_0^T | \langle {}^R D^\mu v, \eta_v \rangle | dt \leq 5 \| v \|_J (\omega_{2-\mu}(T) \| v(0) \| + \int_0^T \omega_{2-\mu}(T-s) \| v'(s) \| ds).$$

On the other hand, noting that

$$\int_0^T \langle {}^R D^\mu \Pi_k v, v \rangle dt = \langle I^{1-\mu} \Pi_k v(T), v(T) \rangle - \int_0^T \langle I^{1-\mu} \Pi_k v, v' \rangle dt,$$

and hence, by the Cauchy-Schwarz inequality and the first inequality in (14),

$$\begin{aligned} \left| \int_0^T \langle {}^R D^\mu \Pi_k v, v \rangle dt \right| &\leq \|\Pi_k v\|_J \int_0^T \left[\omega_{1-\mu}(T-t) \|v(T)\| + \int_0^t \omega_{1-\mu}(t-s) ds \|v'(t)\| \right] dt \\ &\leq 4 \|v\|_J \left(\|v(T)\| \omega_{2-\mu}(T) + \int_0^T \omega_{2-\mu}(t) \|v'(t)\| dt \right). \end{aligned}$$

We combine the above two inequalities and use that $\|v\|_J \leq \|v(0)\| + \int_0^T \|v'\| dt$, we obtain the bound below that will be used to show the convergence of our scheme,

$$\left| \int_0^T \langle {}^R D^\mu v, \eta_v \rangle dt \right| + \left| \int_0^T \langle {}^R D^\mu \Pi_k v, v \rangle dt \right| \leq C T^{1-\mu} \left(\|v(T)\| + \int_0^T \|v'\| dt \right)^2. \quad (17)$$

5 Error estimates

This section is devoted to investigate the convergence of the DG FE scheme, (7). We decompose the error as follows:

$$U - u = \zeta + \Pi_k \xi + \eta_u \quad \text{with} \quad \zeta = U - \Pi_k R_h u. \quad (18)$$

Recall that $\xi = R_h u - u$ and $\eta_u = \Pi_k u - u$. The main task now is to estimate ζ .

Theorem 2 Choose $U^0 = R_h u_0$. For $1 \leq n \leq N$, we have

$$\|\zeta\|_{L^2(H^1)}^2 \leq C \left(h^4 C_1(k, u) + C_2(k, u) + h^2 k^2 \|u\|_{L^2(H^2)}^2 \right),$$

where

$$\begin{aligned} C_1(k, u) &= \max_{n=1}^N \left(k_n^{-\frac{\mu}{2}} \int_{I_n} \|u'\|_2 dt \right)^2 + \left(\|u_0\|_2 + \|u'\|_{L^1(H^2)} \right)^2, \\ C_2(k, u) &= \max_{n=1}^N k_n^{2\ell-2-\mu} \left(k_n^{-\mu} \left(\int_{I_n} \left\| \frac{\partial^\ell u}{\partial t^\ell} \right\| dt \right)^2 + \left(\int_{I_n} \left\| \nabla \frac{\partial^\ell u}{\partial t^\ell} \right\| dt \right)^2 \right). \end{aligned} \quad (19)$$

Proof We start our proof by taking the inner product of (1) with ζ , using the identity ${}^c D^\mu u(t) = {}^R D^\mu u(t) - \omega_{1-\mu}(t) u_0$, and then integrating over the time subinterval I_n ,

$$\int_{I_n} [\langle {}^R D^\mu u, \zeta \rangle + a(t, u, \zeta)] dt = \int_{I_n} \langle f + \omega_{1-\mu}(t) u_0, \zeta \rangle dt.$$

The above equation, the DG FE scheme (7) and the decomposition in (18) imply

$$\begin{aligned} \int_0^T \left(\langle {}^R D^\mu \zeta, \zeta \rangle + |\zeta|_1^2 \right) dt &= \int_0^T \langle \omega_{1-\mu}(t) \xi(0), \zeta \rangle dt \\ &\quad - \int_0^T \left[\langle {}^R D^\mu (\Pi_k \xi + \eta_u), \zeta \rangle + a(t, \Pi_k \xi + \eta_u, \zeta) \right] dt. \end{aligned} \quad (20)$$

Now, using the continuity property in Lemma 2 (ii), we notice that

$$\begin{aligned} \left| \int_0^T \langle {}^R D^\mu \eta_u, \zeta \rangle dt \right| &\leq C \int_0^T \langle {}^R D^\mu \eta_u, \eta_u \rangle dt + \frac{1}{4} \int_0^T \langle {}^R D^\mu \zeta, \zeta \rangle dt, \\ \left| \int_0^T \langle {}^R D^\mu \Pi_k \xi, \zeta \rangle dt \right| &\leq C \int_0^T \langle {}^R D^\mu \Pi_k \xi, \Pi_k \xi \rangle dt + \frac{1}{4} \int_0^T \langle {}^R D^\mu \zeta, \zeta \rangle dt. \end{aligned}$$

In addition, following the steps in (9), we observe

$$\int_0^T \langle \omega_{1-\mu}(t) \xi(0), \zeta \rangle dt \leq \frac{1}{4} \int_0^T \langle {}^R D^\mu \zeta, \zeta \rangle dt + C T^{1-\mu} \|\xi(0)\|^2$$

Inserting the above three inequalities in (20), then simplifying, and using the positivity property of ${}^R D^\mu$, Lemma 2 (i), yield

$$\begin{aligned} \int_0^T |\zeta|_1^2 dt &\leq C T^{1-\mu} \|\xi(0)\|^2 + C \int_0^T \left(\langle {}^R D^\mu \eta_u, \eta_u \rangle + \langle {}^R D^\mu \Pi_k \xi, \Pi_k \xi \rangle \right) dt \\ &\quad + \sum_{n=1}^N \left| \int_{I_n} a(t, \Pi_k \xi + \eta_u, \zeta) dt \right|. \end{aligned} \quad (21)$$

From the definitions of the time projection Π_k and the space projection R_h ,

$$\int_{I_n} \langle \mathcal{A}(t_n) \nabla(\Pi_k \xi + \eta_u), \nabla \zeta \rangle dt = \int_{I_n} \langle \mathcal{A}(t_n) \nabla \xi, \nabla \zeta \rangle dt = \int_{I_n} \langle [\mathcal{A}(t_n) - \mathcal{A}(t)] \nabla \xi, \nabla \zeta \rangle dt$$

and so,

$$\begin{aligned} \left| \int_{I_n} a(t, \Pi_k \xi + \eta_u, \zeta) dt \right| &= \left| \int_{I_n} \langle \mathcal{A}(t_n) \nabla(\Pi_k \xi + \eta_u) + [\mathcal{A}(t) - \mathcal{A}(t_n)] \nabla(\Pi_k \xi + \eta_u), \nabla \zeta \rangle dt \right| \\ &= \left| \int_{I_n} \langle [\mathcal{A}(t) - \mathcal{A}(t_n)] \nabla(\eta_\xi + \eta_u), \nabla \zeta \rangle dt \right| \\ &\leq C k_n \int_{I_n} \|\nabla(\eta_\xi + \eta_u)\| \|\nabla \zeta\| dt. \end{aligned}$$

Thus, by the inequality $\|\nabla \eta_\xi(t)\| \leq \|\nabla \xi(t)\| + 4k_n^{-1} \int_{I_n} \|\nabla \xi(s)\| ds$ (follows from the triangle inequality and the first property of Π_k in (14)) for $t \in I_n$, and property (5),

$$\left| \int_{I_n} a(t, \Pi_k \xi + \eta_u, \zeta) dt \right| \leq C k_n^2 \int_{I_n} (\|\nabla \xi\|^2 + \|\nabla \eta_u\|^2) dt + \frac{1}{2} \int_{I_n} |\zeta|_1^2 dt.$$

Inserting this in (21) and using (11) for $t = 0$, we get

$$\begin{aligned} \int_0^T |\zeta|_1^2 dt &\leq C h^4 \|u_0\|_2^2 \\ &\quad + C \sum_{n=1}^N \int_{I_n} \left(\langle {}^R D^\mu \eta_u, \eta_u \rangle + \langle {}^R D^\mu \Pi_k \xi, \Pi_k \xi \rangle + k_n^2 (\|\nabla \xi\|^2 + \|\nabla \eta_u\|^2) \right) dt. \end{aligned}$$

But, for $t \in I_n$ and for $\ell \in \{1, 2\}$,

$$\begin{aligned} \int_{I_n} \langle {}^R D^\mu \eta_u, \eta_u \rangle dt &\leq C k_n \max_{j=1}^n k_j^{2\ell-2-\mu} \left(\int_{I_j} \left\| \frac{\partial^\ell u}{\partial t^\ell} \right\| dt \right)^2 \quad \text{by Lemma 3,} \\ \|\nabla \xi(t)\| &\leq C h \|u(t)\|_2 \quad \text{by the elliptic projection error (11),} \\ \|\nabla \eta_u(t)\| &\leq C k_n^{\ell-1} \int_{I_n} \left\| \nabla \frac{\partial^\ell u}{\partial t^\ell} \right\| ds \quad \text{by the time projection error (15),} \end{aligned}$$

where in the first inequality we also used the non-increasing time step assumption. So,

$$\begin{aligned} \int_0^T |\zeta|_1^2 dt &\leq C h^4 \|u_0\|_2^2 + C \int_0^T \langle {}^R D^\mu \Pi_k \xi, \Pi_k \xi \rangle dt + C h^2 k^2 \int_0^T \|u\|_2^2 dt \\ &\quad + C \max_{n=1}^N k_n^{2\ell-2-\mu} \left(\left(\int_{I_n} \left\| \frac{\partial^\ell u}{\partial t^\ell} \right\| dt \right)^2 + k_n^\mu \left(\int_{I_n} \left\| \nabla \frac{\partial^\ell u}{\partial t^\ell} \right\| dt \right)^2 \right). \end{aligned} \quad (22)$$

It remains to estimate $\int_0^T \langle {}^R D^\mu \Pi_k \xi, \Pi_k \xi \rangle dt$. From the decomposition:

$$\int_{I_n} \langle {}^R D^\mu \Pi_k \xi, \Pi_k \xi \rangle dt = \int_{I_n} \left[\langle {}^R D^\mu \eta_\xi, \eta_\xi \rangle + \langle {}^R D^\mu \xi, \eta_\xi \rangle + \langle {}^R D^\mu \Pi_k \xi, \xi \rangle \right] dt. \quad (23)$$

By Lemma 3,

$$\int_{I_n} \langle {}^R D^\mu \eta_\xi, \eta_\xi \rangle dt \leq C k_n^{1-\mu} \max_{j=1}^n \left(\int_{I_j} \|\xi'\| dt \right)^2 \leq C k_n \max_{j=1}^n \left(k_j^{-\frac{\mu}{2}} \int_{I_j} \|\xi'\| dt \right)^2.$$

Inserting the above bound in (23), then summing over n and using the achieved bound in (17), we obtain

$$\int_0^T |\langle {}^R D^\mu \Pi_k \xi, \Pi_k \xi \rangle| dt \leq C \max_{n=1}^N \left(k_n^{-\frac{\mu}{2}} \int_{I_n} \|\xi'\| dt \right)^2 + C \left(\|\xi(0)\| + \int_0^T \|\xi'\| dt \right)^2. \quad (24)$$

Finally, to complete the proof, we combine (22) and (24). \square

In the next theorem we show our main convergence results of the DG FE solution. Typically, the exact solution u of problem (1) satisfies the finite regularity assumptions:

$$\|u'(t)\|_2 + t \|u''(t)\|_1 \leq \mathbf{M} t^{\sigma-1} \quad \text{for } t > 0, \quad (25)$$

for some positive constants \mathbf{M} and σ . Due to the singular behaviour u near $t = 0$, we employ a family of non-uniform meshes, where the time-steps are graded towards $t = 0$; see [19, 23]. More precisely, for a fixed parameter $\gamma \geq 1$, we assume that

$$t_n = (n/N)^\gamma T \quad \text{for } 0 \leq n \leq N. \quad (26)$$

One can easily see that the sequence of time-step sizes $\{k_j\}_{j=1}^N$ is nondecreasing, that is, $k_i \leq k_j$ for $1 \leq i \leq j \leq N$. One can also show the following mesh property:

$$k_j \leq \gamma k t_j^{1-1/\gamma}. \quad (27)$$

Theorem 3 *Let u be the solution of (1) satisfying the regularity property (25) with $\sigma > \mu/2$. Let U be the DG FE solution defined by (7). Then, we have*

$$\int_0^T \|U - u\|^2 dt \leq C(h^4 + k^{\gamma(2\sigma-\mu)}) \quad \text{for } 1 \leq \gamma \leq \frac{4-\mu}{2\sigma-\mu}$$

where C is a constant that depends on T , μ , γ , σ , and on \mathbf{M} .

Proof From the decomposition of the error in (18), the triangle inequality, the bound in Theorem 2, the inequality $\|\Pi_k \xi\|_{L^2(L^2)} \leq \|\xi\|_{L^2(L^2)}$ by (14), the elliptic projection error (11), the error from the time projection (15), we have

$$\int_0^T \|U - u\|^2 dt \leq C(h^4 C_1(k, u) + C_2(k, u) + h^2(h^2 + k^2)\|u\|_{L^2(H^2)}^2).$$

By the definitions of $C_1(k, u)$ and $C_2(k, u)$ in (19), the regularity assumption (25), and the inequality $h^2 k^2 \leq \frac{1}{2}(h^4 + k^4)$, we observe

$$\begin{aligned} \int_0^T \|U - u\|^2 dt &\leq C h^4 \max_{n=1}^N \left(k_n^{-\frac{\mu}{2}} \int_{I_n} t^{\sigma-1} dt \right)^2 + C h^4 \left(1 + \int_0^T t^{\sigma-1} dt \right)^2 \\ &\quad + C k_1^{-\mu} \left(\int_{I_1} t^{\sigma-1} dt \right)^2 + C \max_{n=2}^N k_n^{2-\mu} \left(\int_{I_n} t^{\sigma-2} dt \right)^2 + C h^2 k^2 \\ &\leq C(h^4 \max_{n=1}^N k_n^{2\sigma-\mu} + h^4 + k_1^{2\sigma-\mu} + \max_{n=2}^N k_n^{4-\mu} t_n^{2\sigma-4} + k^4) \\ &\leq C(h^4 + k^{\min\{\gamma(2\sigma-\mu), 4-\mu\}}) \end{aligned}$$

where in the last inequality, by the mesh property (27), we used

$$k_n^{4-\mu} t_n^{2(\sigma-2)} \leq C k^{4-\mu} t_n^{2(\sigma-2)+4-\mu-(4-\mu)/\gamma} \leq C k^{\min\{\gamma(2\sigma-\mu), 4-\mu\}}. \quad \square$$

6 Numerical results

We present a sample of numerical tests using a model problem in one space dimension, of the form (1) with $\Omega = (0, 1)$, $[0, T] = [0, 1]$, and $\mathcal{A}(x, t) = 1 + t^{3/2}$. We choose $u_0(x) = \sin(\pi x)$ for the initial data and choose the source term f so that

$$u(t) = (1 + t^{1-\mu}) \sin(\pi x). \quad (28)$$

One easily verifies that the regularity condition (25) holds for $\sigma = 1 - \mu$.

The numerical tests below reveal faster rates of convergence than those suggested by Theorem 3, and that our regularity assumptions are more restrictive than is needed in practice. More precisely, Theorem 3 shows suboptimal (in time) convergence of order $O(k^{2-\frac{\mu}{2}} + h^2)$ for sufficiently graded time meshes in the time-space L^2 -norm. However, we demonstrate numerically optimal (in both time and space) rates of convergence in the stronger $L^\infty(L^2)$ -norm. To this end, We introduce a finer mesh

$$\mathcal{G}^m = \{t_{j-1} + \ell k_j/m : j = 1, 2, \dots, N \text{ and } \ell = 0, 1, \dots, m\}, \quad (29)$$

Table 1 Errors and time convergence rates with $\mu = 0.3$ for various choices of γ .

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
10	5.8997e-03		1.1252e-03		9.9332e-04	
20	3.5981e-03	0.71339	4.1163e-04	1.4507	2.5524e-04	1.9604
40	2.1827e-03	0.72111	1.5008e-04	1.4556	6.4530e-05	1.9838
80	1.3208e-03	0.72468	5.4700e-05	1.4562	1.6137e-05	1.9996
160	7.9804e-04	0.72692	1.9995e-05	1.4519	4.0085e-06	2.0092
320	4.8168e-04	0.72840	7.3478e-06	1.4443	9.9164e-07	2.0152

Table 2 Errors and time convergence rates with $\mu = 0.5$ for various choices of γ .

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$		$\gamma = 4$	
10	1.149e-02		3.262e-03		1.560e-03		1.882e-03	
20	7.641e-03	0.589	1.619e-03	1.011	5.972e-04	1.385	4.869e-04	1.951
40	5.151e-03	0.569	8.037e-04	1.010	2.192e-04	1.446	1.209e-04	2.009
80	3.641e-03	0.500	3.997e-04	1.008	7.867e-05	1.478	2.933e-05	2.044
160	2.570e-03	0.503	1.992e-04	1.005	2.797e-05	1.492	7.011e-06	2.064
320	1.812e-03	0.504	9.940e-05	1.003	9.908e-06	1.497	1.774e-06	1.982

Table 3 Errors and time convergence rates for various choices of γ .

$\mu = 2/3$								
N	$\gamma = 1$		$\gamma = 2$		$\gamma = 4$		$\gamma = 6$	
10	1.677e-02		7.579e-03		3.416e-03		3.261e-03	
20	1.327e-02	0.338	4.677e-03	0.696	1.393e-03	1.294	9.087e-04	1.843
40	1.044e-02	0.346	3.036e-03	0.623	5.553e-04	1.327	2.471e-04	1.879
80	8.191e-03	0.350	1.940e-03	0.646	2.205e-04	1.332	6.435e-05	1.941
160	6.427e-03	0.350	1.229e-03	0.658	8.753e-05	1.333	1.643e-05	1.970
$\mu = 0.7$								
N	$\gamma = 1$		$\gamma = 3$		$\gamma = 5$		$\gamma = 7$	
10	1.792e-02		5.149e-03		3.625e-03		3.991e-03	
20	1.446e-02	0.309	2.905e-03	8.258e-01	1.318e-03	1.459	1.121e-03	1.832
40	1.160e-02	0.318	1.577e-03	8.810e-01	4.673e-04	1.496	3.052e-04	1.877
80	9.290e-03	0.321	8.479e-04	8.955e-01	1.652e-04	1.499	7.981e-05	1.935
160	7.447e-03	0.319	4.547e-04	8.989e-01	5.843e-05	1.500		

and define the discrete maximum norm $\|v\|_{\mathcal{G}^m} = \max_{t \in \mathcal{G}^m} \|v(t)\|$, so that, for sufficiently large values of m , $\|U_h - u\|_{\mathcal{G}^m}$ approximates the uniform error $\|U_h - u\|_{L^\infty(L^2)}$. In all tables, we choose $m = 10$.

For the numerical illustration of the convergence rates in time, we choose M (the number of uniform spatial subintervals) to be sufficiently large such that the spatial error is negligible compared to the error from the time discretization. We employ a time mesh of the form (26). Tables 1, 2 and 3 show the error (in the stronger $L^\infty(L^2)$) and the rates of convergence when $\mu = 0.3, 0.5, 2/3$ and 0.7 respectively, for various choices of N and γ . We observe optimal rates of order $O(k^{\gamma\sigma})$ for various choices of $1 \leq \gamma \leq \frac{2}{\sigma}$ which is faster than the rate $O(k^{\frac{\gamma}{2}(2\sigma-\mu)})$ for $1 \leq \gamma \leq \frac{4-\mu}{2\sigma-\mu}$ predicted by our theory in Theorem 3. Noting that, in Table 3, $\sigma \leq \mu$ and thus the assumption $\sigma > \mu/2$ in this theorem is not sharp.

Next, we test the performance of the spatial FEs discretization of the scheme (7). A uniform spatial mesh that consists of M subintervals where each is of width

Table 4 Errors and convergence rates in space with $\mu = 0.3, 0.5$ and 0.7 .

M	$\mu = 0.3$		$\mu = 0.5$		$\mu = 0.7$	
10	1.2156e-02		1.2780e-02		1.2563e-02	
20	3.1130e-03	1.9653	3.2743e-03	1.9646	3.1768e-03	1.9836
40	7.8803e-04	1.9820	8.2897e-04	1.9818	7.9873e-04	1.9918
80	1.9826e-04	1.9909	2.0864e-04	1.9903	2.0029e-04	1.9956
160	4.9724e-05	1.9954	5.2355e-05	1.9946	5.1065e-05	1.9717

h will be used. We refine the time mesh such that the spatial error is dominating. By Theorem 3, a convergence of order $O(h^2)$ is expected. We illustrate these results in Table 4.

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